ALGEBRAIC LOGARITHMIC DEFORMATIONS AND APPLICATIONS TO SMOOTHINGS OF FANO VARIETIES WITH NORMAL CROSSING SINGULARITIES.

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Abstract. In this paper we first develop, following Kawamata and Namikawa, a logarithmic deformation theory for algebraic varieties over any field k and we obtain criteria for a Fano variety X defined over an algebraically closed field of characteristic zero with normal crossing singularities to be smoothable. In particular, we show that if $T^1(X) \cong \mathcal{O}_D$, where D is the singular locus of X, then X is smoothable.

1. Introduction

In this paper we study the deformation theory of a Fano variety defined over an algebraically closed field of characteristic zero with normal crossing singularities. In particular we investigate when such a variety is smoothable. By this we mean that there is a flat projective morphism $f\colon \mathcal{X} \to \Delta$, where Δ is the spectrum of a discrete valuation ring (R, m_R), such that $\mathcal{X} \otimes_R (R/m_R) \cong X$ and $\mathcal{X} \otimes_R K(R)$ is smooth over the function field $K(R)$ of R. Moreover, we study when such a smoothing exists with smooth total space $\mathcal{X}$. In this case we say that X is totally smoothable.

Normal crossing singularities appear quite naturally in any degeneration problem. Let $f\colon \mathcal{X} \to C$ be a flat projective morphism from a variety $\mathcal{X}$ to a curve C. Then, according to Mumford's semistable reduction theorem [KKMS73], after a finite base change and a birational modification the family can be brought to standard form $f'\colon \mathcal{X}' \to C'$, where $\mathcal{X}'$ is smooth and the special fibers are simple normal crossing varieties.

Smoothings of Fano varieties play a fundamental role in higher dimensional birational geometry as well. The outcome of the minimal model program starting with a smooth n-dimensional projective variety X, is a $\mathbb{Q}$-factorial terminal projective variety Y such that either K_Y is nef, or Y has a Mori fiber space structure. This means that there is a projective morphism $f\colon Y \to Z$ such that $-K_Y$ f-ample, Z is normal and $\dim Z \leq \dim X - 1$. Suppose that the second case happens and $\dim Z = 1$. Let $z \in Z$ and $Y_z = f^{-1}(z)$. Then Y_z is a Fano variety of dimension $n-1$ and Y is a smoothing of Y_z. The singularities of the special fibers are difficult to describe but normal crossing singularities naturally occur and are the simplest possible non-normal singularities.

Moreover, the study of smoothings $f\colon \mathcal{X} \to \Delta$ such that $\mathcal{X}$ is smooth, $-K_{\mathcal{X}}$ is f-ample and the special fiber is a simple normal crossing divisor, has a central role in the classification of smooth Fano varieties [Fu90]. In dimension two T.

2000 *Mathematics Subject Classification.* Primary 14D15, 14D06, 14J45.
Key words and phrases. Algebraic geometry.

Fujita [Fu90] has described all the possible degenerations of smooth Del Pezzo surfaces to simple normal crossing Del Pezzo surfaces and Y. Kachi showed that all these actually occur [Kac07]. As far as we know this problem is completely open in higher dimensions.

It is therefore of interest to study which Fano varieties with normal crossing singularities are smoothable and in particular, which are totally smoothable.

The paper is organized as follows.

In section 3 we extend the theory of logarithmic deformations developed by Y. Kawamata and N. Namikawa [KawNa94] for complex analytic spaces with normal crossing singularities, to the case of algebraic varieties. An algebraic theory of logarithmic structures was previously defined by K. Kato [Ka88]. However, the one defined by Kawamata and Namikawa is more geometric and in my opinion has the advantage of making it easier to calculate the various obstruction spaces to deformations. This is the reason that I feel it is desirable to have an algebraic version of it. In Definition 3.1 we define the notion of logarithmic structures for an algebraic variety with normal crossing singularities and in Definition 3.10 we define the notion of logarithmic deformations of a variety X with a logarithmic structure $\mathcal{U}$. The passage from the complex analytic case to the algebraic one is by using the étale topology and Artin approximation [Art69]. All the results obtained by Kawamata and Namikawa in the complex analytic case, hold in the algebraic as well. We state the most important ones, we prove those where there is a difference between the complex analytic case and the algebraic one and when there is no difference, we refer the reader to [KawNa94] for the details. In Proposition 3.8 we show that a variety X with normal crossing singularities has a logarithmic structure if and only if $T^1(X) = \mathcal{O}_D$, where D is the singular locus of X. If X is simple normal crossing, i.e., has smooth irreducible components, the condition $T^1(X) \cong \mathcal{O}_D$ is equivalent to Friedman's d-semistability condition [Fr83] and hence Proposition 3.8 is a generalization of [KawNa94, Proposition1.1]. The advantage of considering logarithmic deformations instead of the usual ones, is that locally they behave like deformations of smooth varieties. In fact, in Proposition 3.16 it is shown that any logarithmic deformation $(X_A, \mathcal{U}_A)$ of a variety X with a logarithmic structure $\mathcal{U}$ over a finite Artin k-algebra A has a unique lifting to any finite Artin k-algebra B that is a small extension of A. This makes the logarithmic deformation theory similar to the one of smooth varieties. Based on this fact, exactly as in [KawNa94], Theorem 3.17 describes the tangent space and the obstruction to logarithmic deformations.

In section 4 we study the obstruction spaces to deform a Fano variety X with normal crossing singularities defined over an algebraically closed field of characteristic zero. It is well known that $H^2(T_X)$ and $H^1(T^1(X))$ are obstruction spaces to deformations of X. If X has a logarithmic structure $\mathcal{U}$, Theorem 3.17 shows that $H^2(T_X(log))$ is an obstruction space to logarithmic deformations. If X is simple normal crossing, which means that X has smooth irreducible components, then its obstruction theory is deeply clarified by the work of Friedman [Fr83]. However, in the general case when X is not necessarily reducible, Friedman's theory does not directly apply. In Theorem 4.7 we show that if X is a Fano variety with normal crossing singularities then $H^2(T_X) = 0$. Moreover, if X has a logarithmic structure $\mathcal{U}$, then $H^2(T_X(log)) = 0$ and hence X has unobstructed logarithmic deformations. Usual deformations can be obstructed since the other obstruction space $H^1(T^1(X))$ may not vanish. This is the case in example 6.2. However, $T^1(X)$ is a line bundle

on the singular locus D of X and in order for X to be smoothable one has to impose some positivity conditions on $T^1(X)$ that will force it to vanish. If X has at worst double points, then in Theorem 4.9 we show that $H^1(T^(X)) = 0$ and hence X unobstructed deformations in this case.

In sections 5 we apply the deformation theory developed in the previous sections to obtain criteria for the existence of a smoothing of a Fano variety X with normal crossing singularities. We also study the problem of when X is totally smoothable. Proposition 5.1 shows that if X is totally smoothable, then $T^1(X) \cong \mathcal{O}_D$, where D is the singular locus of X. Therefore by Proposition 3.8, X has a logarithmic structure and the theory of logarithmic deformations applies in this case. The main result of sections 5 is the following.

Theorem 1.1. *Let X be a Fano variety defined over an algebraically closed field of characteristic zero with normal crossing singularities. Assume that one of the following conditions hold:*

(1) *$T^1(X)$ is finitely generated by global sections and that $H^1(T^1(X)) = 0$.*
(2) *X has at worst double point normal crossing singularities and that $T^1(X)$ is finitely generated by global sections.*
(3) *X is d-semistable, i.e., $T^1(X) \cong \mathcal{O}_D$, where D is the singular locus of X.*

Then X is smoothable. Moreover, X is smoothable by a flat deformation $f \colon \mathcal{X} \to \Delta$ such that $\mathcal{X}$ is smooth, if and only if X is d-semistable.

I do not know if the condition that $T^1(X)$ is finitely generated by its global sections is a necessary condition too for X to be smoothable. X is certainly not smoothable of $H^0(T^1(X)) = 0$ [Tz10]. In all the cases of the previous theorem the condition finitely generated by global sections implies that $\mathrm{Def}(X)$ is smooth. If it is true that $\mathrm{Def}(X)$ is smooth for any X, then X smoothable implies that $T^1(X)$ is finitely generated by its global sections too.

In section 6 we give an example of a smoothable and one of a non-smoothable Fano threefold.

Finally, the requirement that we work over an algebraically closed field is more technical than essential. In the general case I believe that the definition of normal crossing singularities must be modified to allow singularities like $x_0^2 + x_1^2 = 0$ in $\mathbb{R}^2$. This would make the arguments more complicated without adding anything of essence to the proofs. This is explained in Remarks 3.2. However, the characteristic zero assumption is essential since we make repeated use of the Akizuki-Kodaira-Nakano vanishing theorem.

2. Terminology-Notation.

All schemes in this paper are defined over an algebraically closed field k.

A reduced scheme X of finite type over k is said to have normal crossing (n.c.) singularities at a point $P \in X$ if $\hat{\mathcal{O}}_{X,P} \cong k(P)[[x_0, \dots, x_n]]/(x_0 \cdots x_r)$, for some $r = r(P)$, where $k(P)$ is the residue field of $\mathcal{O}_{X,P}$ and $\hat{\mathcal{O}}_{X,P}$ is the completion of $\mathcal{O}_{X,P}$ at its maximal ideal. X is called a normal crossing variety if it has normal crossing singularities at every point. In addition, if X has smooth irreducible components then it is called a simple normal crossing variety (s.n.c.).

A reduced projective scheme X with normal crossing singularities is called a Fano variety if and only if ω_X^{-1} is an ample invertible sheaf on X.

For any scheme X we denote by $T^1(X)$ the sheaf of first order deformations of X [Sch68]. If X is reduced then $T^1(X) = \mathcal{E}xt^1_X(\Omega_X, \mathcal{O}_X)$.

A variety X with normal crossing singularities is called d-semistable if and only if $T^1(X) \cong \mathcal{O}_D$, where D is the singular locus of X. If X is a s.n.c. variety defined over $\mathbb{C}$, then by Lemma 3.6 and [Tz09],

$$(I_{X_1}/I_{X_1}I_D) \otimes \cdots \otimes (I_{X_k}/I_{X_k}I_D) = \mathcal{H}om_D(T^1(X), \mathcal{O}_D) \cong \mathcal{O}_D$$

and hence X is d-semistable in the sense of Friedman [Fr83].

We say that X is smoothable if there is a flat morphism of finite type $f \colon \mathcal{X} \to \Delta$, $\Delta = \mathrm{Spec}(R)$, where R is a discrete valuation ring, such that the cental fiber $\mathcal{X}_0$ is isomorphic to X and the general fiber $\mathcal{X}_g$ is smooth over the function field $K(R)$ of R.

Finally, we will repeatedly make use of the Akizuki-Kodaira-Nakano vanishing theorem and its logarithmic version, which we state next.

Theorem 2.1 (Akizuki-Kodaira-Nakano [AN54], [EV92])**.** *Let X be a smooth variety and $\mathcal{L}$ an ample invertible sheaf on X. Then*

$$H^b(\Omega^a_X \otimes \mathcal{L}^{-1}) = 0$$

for all a, b such that $a + b < \dim X$.
Moreover, if D is a reduced simple normal crossings divisor of X, then

$$H^b(\Omega^a_X(\log(D)) \otimes \mathcal{L}^{-1}) = 0$$

for all a, b such that $a + b < \dim X$.

3. Logarithmic Structures.

Let X be a complex analytic space with normal crossing singularities. Y. Kawamata and Y. Namikawa [KawNa94] have defined the notions of logarithmic structures and logarithmic deformations of X. In this section we extend their results to algebraic varieties with normal crossing singularities defined over an arbitrary algebraically closed field k.

Definition 3.1. Let X be a scheme of finite type over an algebraically closed field k of dimension n with at worst normal crossing singularities. Let D be the singular locus of X. A logarithmic atlas on X is a collection

$$\mathcal{U} = \{(U_\lambda, f_\lambda); z_0^{(\lambda)}, \ldots, z_n^{(\lambda)}\}$$

such that

(1) $f_\lambda \colon U_\lambda \to X$ is étale and $U_\lambda, V_\lambda = f_\lambda(U_\lambda)$ are affine.
(2) $\cup_\lambda V_\lambda$ contains the singular locus D of X
(3) $z_i^{(\lambda)} \in \mathcal{O}_{U_\lambda}$. Moreover, there are étale morphisms

$$g_\lambda \colon U_\lambda \to W_\lambda = \mathrm{Spec} \frac{k[x_0, \ldots, x_n]}{(x_0 \cdots x_{r(\lambda)})}$$

with the following properties. Let $W_{\lambda,i}$ be the irreducible component of W_λ given by $x_i = 0$ and $U_{\lambda,i} = g_\lambda^{-1}(W_{\lambda,i})$, $0 \le i \le r(\lambda)$. Then $\mathcal{I}_{U_{\lambda,i}} = (z_i^{(\lambda)})$, $0 \le i \le r(\lambda)$, and $z_i^{(\lambda)} \in \mathcal{O}^*_{U_\lambda}$ for $r(\lambda) < i \le n$. In particular, U_λ is a s.n.c. variety.

(4) Let λ, μ be such that $U_{\lambda\mu} = U_\lambda \times_X U_\mu \neq \emptyset$ and let $p_\lambda \colon U_\lambda \times_X U_\mu \to U_\lambda$, $p_\mu \colon U_\lambda \times_X U_\mu \to U_\mu$, be the projection maps. Then for any connected component Z of the singular locus of $U_{\lambda\mu}$, there is an open neighborhood W of Z in $U_{\lambda\mu}$, a $\sigma \in S_{n+1}$ and invertible functions $u_i^{(\lambda\mu)} \in \mathcal{O}_W^*$ such that

$$p_\lambda^*(z_{\sigma(i)}^{(\lambda)})|_W = u_i^{(\lambda\mu)} p_\mu^*(z_i^{(\mu)})|_W \quad \text{and} \quad u_o^{(\lambda\mu)} \cdots u_n^{(\lambda\mu)} = 1$$

Two log atlases $\mathcal{U}$ and $\mathcal{U}'$ on a variety X with normal crossing singularities are called equivalent if and only if their union is a log atlas on X. An equivalence class of log atlases is called a logarithmic structure (or log structure).

Remarks 3.2. (1) Our definition of a logarithmic structure is a natural generalization of the one given by Kawamata and Namikawa [KawNa94] in the complex analytic case. In that case, the functions $z_i^{(\lambda)}$ are defined by choosing a local isomorphism of an analytic neighborhood U_λ with an open neighborhood of $x_0 \cdots x_N = 0$ in $\mathbb{C}^{n+1}$, where $n = \dim X$ and pulling back the coordinate functions x_i. In the algebraic case there may not be such an isomorphism in the Zariski topology. However, this is remedied by passing to the étale topology and using Artin's algebraization results [Art69].

(2) The reason that in 3.1.4 we require the relations to hold in a neighborhood of any connected component of the singular locus of $U_{\lambda\mu}$ and not on the whole $U_{\lambda\mu}$ is the following. The functions $z_i^{(\lambda)}$ generate the ideal sheaf of a smooth irreducible component $U_{\lambda,i}$ of U_λ. Then $p_\lambda^*(z_i^{(\lambda)})$ generate the ideal sheaf of $p_\lambda^{-1}(U_{\lambda,i})$ and $U_{\lambda\mu} = \cup_i p_\lambda^{-1}(U_{\lambda,i}) = \cup_j p_\mu^{-1}(U_{\mu,j})$. If $p_\lambda^{-1}(U_{\lambda,i})$ and $p_\mu^{-1}(U_{\mu,j})$ were irreducible, then for any i there would be a j such that $p_\lambda^{-1}(U_{\lambda,i}) = p_\mu^{-1}(U_{\mu,j})$ and then it would be reasonable to require 3.1.4 to hold in $U_{\lambda\mu}$. However, it is possible that $p_\lambda^{-1}(U_{\lambda,i})$ and $p_\mu^{-1}(U_{\mu,j})$ are disconnected and their components could in principle mix making a relation as in 3.1.4 unlikely in the whole $U_{\lambda\mu}$. However, if $f_\lambda \colon U_\lambda \to X$ is an étale neighborhood of D (i.e., $f_\lambda^{-1}(D) \cong D$), then the previous complication does not happen and we could require 3.1.4 to hold on all $U_{\lambda\mu}$. Unfortunately though, I do not know if such neighborhoods exist in general. In any case this is just a technicallity that would simplify the definition of logarithmic structures but does not alter the theory in any significant way.

(3) If the base field is not algebraically closed, then I believe that the definition of a normal crossing singularity must be modified. A point $P \in X$ should be called a normal crossing singularity if and only if

$$\widehat{\mathcal{O}}_{X,P} \otimes_L \overline{L} \cong \frac{\overline{L}[x_0, \ldots, x_n]}{(x_0 \cdots x_r)}$$

where L is a coefficient field of $\hat{\mathcal{O}}_{X,P}$ and $\overline{L}$ its algebraic closure. This way singularities like $x_0^2 + x_1^2 = 0$ in $\mathbb{R}^n$ are normal crossing. We believe that the methods used in this paper work in the general case as well. However, the arguments would be more complicated since one has to keep track of the fields involved and worry about separability conditions. For this reason and for the sake of clarity we work over an algebraically closed field.

Next we present some basic properties of logarithmic structures and logarithmic deformations. They are generalization in the algebraic category of the corresponding properties in the complex analytic case [KawNa94]. We will explain the

differences between the two cases and whenever there is none or the proofs are straightforward, we refer the reader to [KawNa94] for the details.

Next we present the main technical tools. The bridge between the analytic case and the algebraic one is provided by Artin's algebraization theorem.

Theorem 3.3 ([Art69]). *Let X_1, X_2 be S-schemes of finite type, and let $x_i \in X_i$, $i = 1, 2$, be points. If $\hat{\mathcal{O}}_{X_1,x_1} \cong \hat{\mathcal{O}}_{X_2,x_2}$, then X_1 and X_2 are locally isomorphic for the étale topology, i.e., there is a common étale neighborhood (X', x') of (X_i, x_i), $i = 1, 2$. This means there is a diagram of étale maps*

$$\begin{tikzcd} & X' \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ X_1 & & X_2 \end{tikzcd}$$

such that $f_1(x') = x_1$, $f_2(x') = x_2$ and inducing an isomorphism of residue fields $k(x_1) \cong k(x') \cong k(x_2)$

We will also need the following technical results.

Lemma 3.4. *Let $f\colon X \to Y$ be an étale map with X affine and Y separated. Then there is an affive cover $\{V_i\}$ of Y such that $U_i = f^{-1}(V_i)$ is affine too.*

Proof. By Zariski's main theorem [Mil80, Theorem 1.8] there is a factorization

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & X' \arrow[dl, "g"] \\ Y & \end{tikzcd}$$

where i is an open immersion and g is finite. Let $V \subset Y$ be open affine. Then $g^{-1}(V)$ is affine and hence since Y is separated, $U = X \cap g^{-1}(V)$ is affine too. The result now follows immediately. □

Lemma 3.5. *Let A be a reduced Noetherian ring such that*

(1) *The minimal primary decomposition of (0) is of the form*

$$(0) = (a_1) \cap \cdots \cap (a_k)$$

(2) *$A/(a_i)$ is regular for all $i = 1, \ldots, k$.*
(3) *For any maximal ideal $m \subset A$,*

$$\widehat{A}_m \cong \frac{(A/m)[[x_1, \ldots, x_n]]}{(x_1 \cdots x_{r(m)})}$$

Let $a \in A$ be an element such that $(a_i) = (a)$, for some $1 \leq i \leq k$. Then there exists $u \in A^*$ a unit such that $a = ua_i$

Proof. Let $a \in A$ such that $(a) = (a_i)$, for some i. Then $a_i = au$, for some $u \in A$. If u is not a unit, then there is a maximal ideal m of A such that $u \in m$. Then by assumption,

$$\widehat{A}_m = \frac{k[[x_1, \ldots, x_n]]}{(x_1 \cdots x_{r(m)})}$$

where $k = A/m$. Moreover, we may take $x_1 = a_i$. Then $(x_1) = (a)$. Let $I_D = (x_1 \cdots \hat{x}_i \cdots x_n; 1 \leq i \leq r(m)) \subset \widehat{A}_m$ be the ideal of the singular locus of $\widehat{A}_m$. Then $(x_1)/(x_1)I_D$ is a free $\widehat{A}_m/I_D$ module. Hence

$$x_1 = ca + a\sum_i g_i x_1 \cdots \hat{x}_i \cdots x_{r(m)}$$

where $c \in k$. Then $x_1 = ua \in m^2$, which is impossible. Hence $u \in A^*$.

□

Lemma 3.6. *Let Y be a smooth variety and $X \subset Y$ a simple normal crossing divisor. Let X_i, $1 \leq i \leq n$ be the irreducible components of X and D its singular locus. Then $T^1(X)$ is an invertible sheaf on D and moreover*

$$\mathcal{H}om_D(T^1(X), \mathcal{O}_D) = \frac{\mathcal{I}_{X_1}}{\mathcal{I}_{X_1}\mathcal{I}_D} \otimes \cdots \otimes \frac{\mathcal{I}_{X_n}}{\mathcal{I}_{X_n}\mathcal{I}_D}$$

Proof. A local calculation shows that indeed $T^1(X)$ is an invertible sheaf on D. Moreover, there is an exact sequence

$$T_Y \otimes \mathcal{O}_X \to \mathcal{N}_{X/Y} \to \mathcal{E}xt^1_X(\Omega_X, \mathcal{O}_X) \to 0.$$

Now since $T^1(X) = \mathcal{E}xt^1_X(\Omega_X, \mathcal{O}_X)$ is invertible on D, it follows that $T^1(X) = \mathcal{N}_{X/Y} \otimes \mathcal{O}_D$ or equivalently that

$$\mathcal{H}om_D(T^1(X), \mathcal{O}_D) = \frac{\mathcal{I}_X}{\mathcal{I}_X^2} \otimes \mathcal{O}_D = \frac{\mathcal{I}_{X_1} \cdots \mathcal{I}_{X_n}}{\mathcal{I}_{X_1}^2 \cdots \mathcal{I}_{X_n}^2 + \mathcal{I}_D \mathcal{I}_{X_1} \cdots \mathcal{I}_{X_n}}$$

Now since $D = \cup_{i_0 < i_1}(X_{i_0} \cap X_{i_1})$, it follows that $\mathcal{I}_D = \prod_{i_0 < i_1}(\mathcal{I}_{X_{i_0}} + \mathcal{I}_{X_{i_1}})$. Then there is a natural surjective map

$$\frac{\mathcal{I}_{X_1}}{\mathcal{I}_{X_1}\mathcal{I}_D} \otimes \cdots \otimes \frac{\mathcal{I}_{X_n}}{\mathcal{I}_{X_n}\mathcal{I}_D} \to \frac{\mathcal{I}_{X_1} \cdots \mathcal{I}_{X_n}}{\mathcal{I}_{X_1}^2 \cdots \mathcal{I}_{X_n}^2 + \mathcal{I}_D \mathcal{I}_{X_1} \cdots \mathcal{I}_{X_n}} = \mathcal{H}om_D(T^1(X), \mathcal{O}_D)$$

A straightforward local calculation shows that for any $1 \leq r \leq n$, $\mathcal{I}_{X_r}/\mathcal{I}_{X_r}\mathcal{I}_D$ is invertible on D and hence

$$\mathcal{H}om_D(T^1(X), \mathcal{O}_D) = \frac{\mathcal{I}_{X_1}}{\mathcal{I}_{X_1}\mathcal{I}_D} \otimes \cdots \otimes \frac{\mathcal{I}_{X_n}}{\mathcal{I}_{X_n}\mathcal{I}_D}$$

as claimed.

□

Remarks 3.7.
1. The previous proposition is true without the assumption that X is a divisor on a smooth variety. However, we only need this version and the above proof is much simpler than the general case.
2. If X is a complex analytic space with normal crossing singularities, then there is an embedding $X \subset Y$ of X as a divisor on a smooth variety Y [Tz09].

Proposition 3.8. *Let X be a scheme of finite type over an algebraically closed field k with normal crossing singularities. Then X admits a logarithmic structure if and only if $T^1(X) \cong \mathcal{O}_D$, where D is the singular locus of X.*

Proof. Suppose that $T^1(X) \cong \mathcal{O}_D$. Let $P \in X$ be a normal crossing singularity. Then by Theorem 3.3, there is a diagram

$$\begin{tikzcd} & U_p \arrow[dl, "f_p"'] \arrow[dr, "g_p"] & \\ X & & W_p \end{tikzcd}$$

where f_p, g_p are étale, $P \in f_p(U_p)$ and

$$W_p = \mathrm{Spec}\frac{k[x_0, \dots, x_n]}{(x_0 \cdots x_{r(p)})}.$$

for some $1 \leq r(p) \leq n$. Moreover, by Lemma 3.4, we can assume that U_p and $f(U_p) = V_p$ are affine. Let $w_i = g_p^*(x_i)$, $1 \leq i \leq r(p)$ and $w_i = 1$, $r(p) < i \leq n$. Repeating this at every point of X we get a collection

$$\mathcal{U} = \{(U_\lambda, f_\lambda); w_0^{(\lambda)}, \dots, w_n^{(\lambda)}\}$$

satisfying the properties 3.1.1, 3.1.2 and 3.1.3. In addition, there are étale maps $g_\lambda \colon U_\lambda \to W_\lambda$, where

$$W_\lambda = \mathrm{Spec}\frac{k[x_0, \dots, x_n]}{(x_0 \cdots x_{r(\lambda)})}.$$

It remains to show that $\mathcal{U}$ can be chosen so that that 3.1.4 is satisfied too. Let $(T^1(X))^* = \mathcal{H}om_D(T^1(X), \mathcal{O}_D)$. Then $(T^1(X))^* \cong \mathcal{O}_D$. Let $(T^1(X))^*_{et}$ be the sheaf in the étale topology induced by $(T^1(X))^*$. Then $(T^1(X))^*_{et} \cong \mathcal{O}_{D_{et}}$ Let $s \in H^0((T^1(X))^*_{et})$ be a nowhere vanishing section and let s_λ its restriction on U_λ. Then since g_λ is étale, it follows that

$$(T^1(U_\lambda))^* = g_\lambda^*((T^1(W_\lambda))^*)$$

Moreover, from Lemma 3.6 it follows that

$$(T^1(W_\lambda))^* = \frac{\mathcal{I}_{W_{\lambda,1}}}{\mathcal{I}_{W_{\lambda,1}}\mathcal{I}_{E_\lambda}} \otimes \cdots \otimes \frac{\mathcal{I}_{W_{\lambda,r(\lambda)}}}{\mathcal{I}_{W_{\lambda,r(\lambda)}}\mathcal{I}_{E_\lambda}}$$

where E_λ is the singular locus of W_λ and $W_{\lambda,i}$ is the irreducible component of W given by $x_i = 0$ for $1 \leq i \leq r(\lambda)$. Hence

$$(T^1(X))^*(U_\lambda) = \frac{\mathcal{I}_{U_{\lambda,1}}}{\mathcal{I}_{U_{\lambda,1}}\mathcal{I}_{D_\lambda}} \otimes \cdots \otimes \frac{\mathcal{I}_{U_{\lambda,r(\lambda)}}}{\mathcal{I}_{U_{\lambda,r(\lambda)}}\mathcal{I}_{D_\lambda}} \tag{3.8.1}$$

where $U_{\lambda,i} = g_\lambda^{-1}(W_{\lambda,i})$, $D_\lambda = g_\lambda^{-1}(E_\lambda)$ is the singular locus of U_λ and $\mathcal{I}_{U_{\lambda,i}}, \mathcal{I}_{D_\lambda}$ the ideal sheaves of $U_{\lambda,i}$ and D_λ in U_λ, respectively. The rest of the argument is mostly as in [KawNa94]. For the sake of completeness, and in order to highlight the differences between the complex analytic and the algebraic case in definition 3.1.3 and 3.1.4, we reproduce it next.

By construction, $\mathcal{I}_{U_{\lambda,i}} = (w_i^{(\lambda)})$, where $w_i^{(\lambda)} = g_\lambda^*(x_i)$, and

$$s_\lambda = s_0^\lambda \otimes \cdots \otimes s_{r(\lambda)}^\lambda$$

where $s_i^\lambda \in \mathcal{I}_{U_{\lambda,i}}/\mathcal{I}_{U_{\lambda,i}}\mathcal{I}_{D_\lambda}$ and hence $s_i^\lambda = \bar{u}_i^\lambda \bar{w}_i^\lambda$ with $\bar{u}_i^\lambda \in \mathcal{O}^*_{D_\lambda \cap U_\lambda}$. If we shrink U_λ we can assume that $u_i^\lambda \in \mathcal{O}^*_{U_\lambda}$ and hence $z_i^{(\lambda)} = u_i^\lambda w_i^\lambda$ generate $\mathcal{I}_{U_{\lambda,i}}$ and

$$s_\lambda = \bar{z}_0^{(\lambda)} \otimes \cdots \otimes \bar{z}_n^{(\lambda)}$$

Let $U_{\lambda\mu} = U_\lambda \times_X U_\mu$, and let $p_\lambda\colon U_{\lambda\mu} \to U_\lambda$, $p_\mu\colon U_{\lambda\mu} \to U_\mu$ be the projection maps. Then

$$\bigcup_i p_\lambda^{-1}(U_{\lambda,i}) = \bigcup_j p_\mu^{-1}(U_{\mu,j}) \tag{3.8.2}$$

Let Z be a connected component of the singular locus of $U_{\lambda\mu}$. Then it is not hard to see that if V is a sufficiently small affine open neighborhood of Z in $U_{\lambda\mu}$, $p_\lambda^{-1}(U_{\lambda,i})\cap V$ and $p_\mu^{-1}(U_{\mu,j})\cap V$ are irreducible and hence (3.8.2) implies that there is $\sigma \in S_{n+1}$ such that $p_\lambda^{-1}(U_{\lambda,\sigma(i)}) \cap V = p_\mu^{-1}(U_{\mu,i}) \cap V$. Then since V is affine, Lemma 3.5 implies that $p_\lambda^*(z^{(\lambda)}_{\sigma(i)})|_V = u_i^{(\lambda\mu)} p_\mu^*(z_i^{(\mu)})|_V$, where $u_i^{(\lambda\mu)} \in \mathcal{O}_V^*$. Then from 3.8.1 in the case of V, it follows that

$$u_0^{(\lambda\mu)} \cdots u_n^{(\lambda\mu)} = 1$$

on Z. Therefore, since V is affine s.n.c., there are $a_j \in H^0(\mathcal{O}_V)$ such that

$$u_0^{(\lambda\mu)} \cdots u_n^{(\lambda\mu)} = 1 + \sum_{j=0}^{n} a_j t_0^{(\mu)} \cdots t_{j-1}^{(\mu)} t_{j+1}^{(\mu)} \cdots t_n^{(\mu)}$$

where $t_i^{(\mu)} = p_\mu^*(z_i^{(\mu)})|_V$. Hence, since $t_0^{(\mu)} \cdots t_n^{(\mu)} = 0$, if we replace each $u_j^{(\lambda\mu)}$ with

$$u_j^{(\lambda\mu)} - a_j t_0^{(\mu)} \cdots t_{j-1}^{(\mu)} t_{j+1}^{(\mu)} \cdots t_n^{(\mu)}$$

we see that condition 3.1.4 is satisfied too. The converse is exactly as in [KawNa94] and we omit it. □

Remark 3.9. If X is a s.n.c. complex analytic space then there is an embedding $X \subset Y$ of X as a divisor in a smooth variety Y [Tz09, Lemma 3.11] and then Lemma 3.6 says that X is d-semistable in the sense of Friedman [Fr83]. Therefore Proposition 3.8 is a generalization of [KawNa94, Proposition 1.1].

Next we define the notion of logarithmic deformations of a n.c. variety X with a logarithmic structure $\mathcal{U}_0$.

Definition 3.10. Let $(X,\mathcal{U}_0)$ be a n.c. algebraic variety with a logarithmic structure. Let $D_1, \dots, D_m$ be the components of the singular locus D of X. Let (A, m_A) be an Artin local k-algebra and $s_1, \dots s_m \in m_A$. Then a logarithmic deformation $(\mathcal{X},\mathcal{U})$ of $(X,\mathcal{U}_0)$ over $\mathcal{A} = (A; s_1, \dots, s_m)$ is a deformation $\mathcal{X}$ of X over A and a logarithmic atlas

$$\mathcal{U} = \{(U_\lambda, f_\lambda); z_0^{(\lambda)}, \dots, z_n^{(\lambda)}\}$$

which is defined as follows:

(1) $f_\lambda\colon U_\lambda \to \mathcal{X}$ are étale and $z_i^{(\lambda)} \in \mathcal{O}_{U_\lambda}$.
(2) The restrictions of f_λ and $z_i^{(\lambda)}$ on X form a logarithmic atlas on X equivalent to $\mathcal{U}_0$.
(3) $z_0^{(\lambda)} \cdots z_n^{(\lambda)} = s_i$, if $f_\lambda(U_\lambda) \cap D_i \neq \emptyset$.
(4) Let λ, μ be such that $U_{\lambda\mu} = U_\lambda \times_X U_\mu \neq \emptyset$ and let $p_\lambda\colon U_\lambda \times_X U_\mu \to U_\lambda$, $p_\mu\colon U_\lambda \times_X U_\mu \to U_\mu$, be the projection maps. Then for any connected component Z of the singular locus of $U_{\lambda\mu} \otimes_A k$, there is an open neighborhood W of Z in $U_{\lambda\mu}$, a $\sigma \in S_{n+1}$ and invertible functions $u_i^{(\lambda\mu)} \in \mathcal{O}_W^*$ such that

$$p_\lambda^*(z^{(\lambda)}_{\sigma(i)})|_W = u_i^{(\lambda\mu)} p_\mu^*(z_i^{(\mu)})|_W \quad \text{and} \quad u_o^{(\lambda\mu)} \cdots u_n^{(\lambda\mu)} = 1$$

Two logarithmic deformations $(\mathcal{X},\mathcal{U})$ and $(\mathcal{X}',\mathcal{U}')$ of $(X,\mathcal{U}_0)$ are called equivalent if and only if there is an A-isomorphism $\phi\colon \mathcal{X} \to \mathcal{X}'$ such that $\mathcal{U}$ and $\phi^*\mathcal{U}'$ define the same log structure in $\mathcal{X}$, i.e., their union is a log atlas on $\mathcal{X}$, where $\phi^*\mathcal{U}'$ is the natural pullback of $\mathcal{U}'$ on $\mathcal{X}$.

Let $\Lambda_m = k[[t_1,\dots,t_m]]$. We shall regard $\mathcal{A}$ as a Λ_m-algebra with the same underlying ring as A whose structure homomorphism $\alpha\colon \Lambda_m \to \mathcal{A}$ is given by $\alpha(t_i) = s_i$. Then the log deformations of $(X,\mathcal{U}_0)$ define a functor

$$LD(X,\mathcal{U}_0)\colon Art_{\Lambda_m}(k) \to (Sets)$$

Next we study the logarithmic deformation theory of a n.c. variety X with a logarithmic structure $\mathcal{U}_0$. We will describe the tangent space of $LD(X,\mathcal{U}_0)$ and obtain obstruction spaces for logarithmic deformations. We follow the steps of the theory developed by Kawamata-Namikawa and we adapt it in the algebraic case. As a first step we define the sheaf of relative logarithmic differentials on $\mathcal{X}$.

Definition-Proposition 3.11. Let $(\mathcal{X},\mathcal{U})$ be a logarithmic deformation of a n.c. variety $(X,\mathcal{U}_0)$ with a logarithmic structure over $\mathcal{A}$. Then the sheaf of relative logarithmic differentials $\Omega_{\mathcal{X}/\mathcal{A}}(log)$ is a locally free $\mathcal{O}_{\mathcal{X}}$-module of rank $n = \dim X$ defined as follows:

(1) Let $f_\lambda\colon U_\lambda \to \mathcal{X}$ be an étale map in $\mathcal{U}$. Then we define

$$\Omega_{U_\lambda/\mathcal{A}}(log) = \frac{\Omega_{U_\lambda/\mathcal{A}} \oplus \left[\oplus_{j=0}^{n}\mathcal{O}_{U_\lambda} e_j^{(\lambda)}\right]}{(\mathrm{d}z_j^{(\lambda)} - z_j^{(\lambda)}e_j^{(\lambda)}, \sum_{k=0}^{n} e_k^{(\lambda)}; j = 0,\dots,n)}$$

(2) Let λ,μ be such that $U_{\lambda\mu} = U_\lambda \times_X U_\mu \neq \emptyset$ and let $p_\lambda\colon U_\lambda \times_X U_\mu \to U_\lambda$, $p_\mu\colon U_\lambda \times_X U_\mu \to U_\mu$, be the projection maps. Let Z be a connected component of the singular locus of $U_{\lambda\mu}$ and W a neighborhood of it as in Definition 3.1. Then on W we define the maps

$$\phi^Z_{\lambda\mu}\colon p_\lambda^*(\Omega_{U_\lambda/\mathcal{A}}(log))|_W \to p_\mu^*(\Omega_{U_\mu/\mathcal{A}}(log))|_W$$

by

$$\phi^Z_{\lambda\mu}(e^{(\lambda)}_{\sigma(j)}) = e_j^{(\mu)} + \frac{\mathrm{d}u_j^{(\lambda\mu)}}{u_j^{(\lambda\mu)}}$$

These maps glue to a map

$$\phi_{\lambda\mu}\colon p_\lambda^*(\Omega_{U_\lambda/\mathcal{A}}(log)) \to p_\mu^*(\Omega_{U_\mu/\mathcal{A}}(log))$$

Now we use Grothendieck's descent theory to show that the sheaves $\Omega_{U_\lambda/\mathcal{A}}(log)$ come from a sheaf $\Omega_{\mathcal{X}/\mathcal{A}}(log)$ on $\mathcal{X}$.

Theorem 3.12 (Fpqc descent, Theorem 4.23 [BLLSNA])**.** *Let X be a scheme and $\mathcal{U}$ a covering sieve in the étale topology generated by a finite number of quasi-compact X-schemes. To give a quasi-coherent module on X is equivalent to give it $\mathcal{U}$-locally. This means to give for any $U \in \mathcal{U}$ a quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}_U$, and for any morphism $\phi\colon V \to U$ in $\mathcal{U}$ an isomorphism $\sigma_\phi\colon \mathcal{F}_V \to \phi^*\mathcal{F}_U$, such that*

given morphisms $W \xrightarrow{\psi} V \xrightarrow{\phi} U$ *in* $\mathcal{U}$*, the following diagram commutes*

$$\begin{tikzcd} \mathcal{F}_W \arrow[rr, "\sigma_{\phi\psi}"] \arrow[dr, "\sigma_\psi"'] & & \psi^*\phi^*\mathcal{F}_U \\ & \psi^*\mathcal{F}_V \arrow[ur, "\sigma_\phi"'] & \end{tikzcd}$$

Finally, We also define

$$T_{\mathcal{X}/\mathcal{A}}(log) = \mathcal{H}om_{\mathcal{X}}(\Omega_{\mathcal{X}/\mathcal{A}}(log), \mathcal{O}_{\mathcal{X}})$$

One can check exactly as in [KawNa94] that the conditions of descent are satisfied and hence there is a coherent sheaf $\Omega_{\mathcal{X}/\mathcal{A}}(log)$ on $\mathcal{X}$ such that

$$f_\lambda^*\Omega_{\mathcal{X}/\mathcal{A}}(log) = \Omega_{U_\lambda/\mathcal{A}}(log)$$

for all λ.

The crucial thing about logarithmic deformations is that locally they behave exactly as if X was smooth. In particular, we will show that if X is affine and $(\mathcal{X}_A,\mathcal{U}_A)$ is a logarithmic deformation of $(X,\mathcal{U}_0)$ over $\mathcal{A}$, then for any square zero extension

$$0 \to J \to \mathcal{B} \to \mathcal{A} \to 0$$

there is a unique lifting of $(\mathcal{X}_A,\mathcal{U}_A)$ to $\mathcal{B}$. This property allows us to work the logarithmic deformation theory in a similar way as the smooth case. We start with some preparatory definitions and results.

Definition 3.13 ([Tz10])**.** Let

$$0 \to J \to B \to A \to 0$$

be a small extension of Artin rings and $X_A \in Def(X)(A)$. Let (X_B^i, ϕ_i), $i = 1, 2$ be pairs where $X_B^i \in Def(X)(B)$ and $\phi_i \colon X_A \to X_B^i \otimes_B A$ isomorphisms. We say that the pair (X_B^1, ϕ_1) is isomorphic to the pair (X_B^2, ϕ_2) if and only if there is a B-isomorphism $\psi\colon X_B^1 \to X_B^2$ such that $\psi\phi_1 = \phi_2$.

We define by $Def(X_A/A, B)$ to be the set of isomorphism classes of pairs $[X_B, \phi]$ of deformations $X_B \in Def(X)(B)$ and marking isomorphisms $\phi\colon X_A \to X_B \otimes_B A$.

Lemma 3.14. *Let X ba a reduced scheme of finite type over a field k and*

$$0 \to k \to B \to A \to 0$$

be a small extension of finite Artin k-algebras. Let X_A be a deformation of X over A. Then there is a one to one map

$$\theta\colon Def(X_A/A, B) \to \mathrm{Ext}^1_{X_A}(\Omega_{X_A}, \mathcal{O}_X)$$

Proof. Let X_B be a deformation of X over B lifting X_A together with an isomorphism $\phi\colon X_A \to X_B \otimes_A B$. Then, since X is reduced, there is a short exact sequence

$$0 \to \mathcal{O}_X \to \Omega_{X_B} \otimes_B A \to \Omega_{X_A} \to 0$$

which gives an element of $\mathrm{Ext}^1_{X_A}(\Omega_{X_A}, \mathcal{O}_X)$. This defines the map θ. Since in general $Def(X_A/A, B)$ does not have a vector space structure over k, θ is just a map of sets. Now similar arguments as in the case when $A = k$ show that θ is one to one [Ser06] . $\square$

Lemma 3.15. *Let X be a reduced affine scheme of finite type over a field k. Suppose there is a commutative diagram*

$$
\begin{tikzcd}
 & Z_A \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\
X_A^1 \arrow[dr, "f_1"'] & & X_A^2 \arrow[dl, "f_2"] \\
 & \mathrm{Spec} A &
\end{tikzcd} \tag{3.15.1}
$$

where X_A^1, X_A^2 are deformations of X over a finite Artin k-algebra A and g_1, g_2 are faithfully flat étale maps. Then $X_A^1 \cong X_A^2$ over A.

Proof. We do induction on the length $l(A)$ of A. If $l(A) = 0$ then there is nothing to show. By the induction hypothesis we may assume that there is a square zero extension

$$0 \to k \to A \to B \to 0$$

and diagram (3.15.1) are two liftings of a faithfully flat étale map $g\colon Z_B \to X_B$, where X_B is a deformation of X over B. By Lemma 3.14, there is a one to one map

$$\theta\colon Def(X_B/B, A) \to \mathrm{Ext}^1_{X_B}(\Omega_{X_B}, \mathcal{O}_X)$$

Let $e_i = \theta(X_A^i)$, $i = 1, 2$. Since g is faithfully flat and étale, there is a natural injection

$$\sigma\colon \mathrm{Ext}^1_{X_B}(\Omega_{X_B}, \mathcal{O}_X) \to \mathrm{Ext}^1_{Z_B}(\Omega_{Z_B}, \mathcal{O}_Z)$$

given by g^*, where $Z = Z_B \otimes_B k$ is étale over X. Now the explicit description of θ given in the proof of Lemma 3.14 show that $\sigma(e_1)$ and $\sigma(e_2)$ are both represented by the extension

$$0 \to \mathcal{O}_Z \to \Omega_{Z_A} \otimes_B A \to \Omega_{Z_B} \to 0$$

and hence $e_1 = e_2$ and therefore $X_A^1 \cong X_A^2$ over A. □

Proposition 3.16. *Let $(X,\mathcal{U})$ be an affine n.c. variety with a logarithmic structure $\mathcal{U} = \{(U, f); z_0, \ldots, z_n\}$ consisting of a single faithfully flat étale cover $U \xrightarrow{f} X$. Suppose that the singular locus D of X is connected. Let A be a finite Artin Λ_1-algebra and let $(X_A,\mathcal{U}_A)$ be a logarithmic deformation of $(X,\mathcal{U})$ over A. Then for any square zero extension*

$$0 \to J \to B \to A \to 0$$

of finite Artin Λ_1-algebras, there exists exactly one lifting $(X_B,\mathcal{U}_B)$ of $(X_A,\mathcal{U}_A)$ over B.

Proof. By induction on the length of B we may assume that $J \cong k$. Suppose $\mathcal{U}_A = \{(U_A, f_A); z_0^A, \ldots, z_n^A\}$. According to the definition of log deformations, $f_A\colon U_A \to X_A$ is étale and $z_0^A \cdots z_n^A = s_A$, where $s_A \in m_A$ is the element defining the Λ_1-algebra structure of A. Suppose there are two liftings $(X_B,\mathcal{U}_B)$, $(\tilde{X}_B,\tilde{\mathcal{U}}_B)$ of $(X_A,\mathcal{U}_A)$. Suppose that $\mathcal{U}_B = \{(U_B, f_B); z_0^B, \ldots, z_n^B\}$ and $\tilde{\mathcal{U}}_B = \{(\tilde{U}_B, \tilde{f}_B); \tilde{z}_0^B, \ldots, \tilde{z}_n^B\}$ of $(X_A,\mathcal{U}_A)$. By the definition of logarithmic deformations, $f_B\colon U_B \to X_B$ and $\tilde{f}_B\colon \tilde{U}_B \to \tilde{X}_B$ are étale liftings of $f_A\colon U_A \to X_A$ over B and moreover, $\tilde{z}_0^B \cdots \tilde{z}_n^B = z_0^B \cdots z_n^B = s_B \in m_B$, where $s_B \in m_B$ defines the Λ_1-algebra

structure of B. By Lemma 3.14, the deformation U_B and $\tilde{U}_B$ correspond to two elements of $\mathrm{Ext}^1_{U_A}(\Omega_{U_A}, \mathcal{O}_U)$. Now for any closed point $P \in U$, there is an injection

$$\mu_P \colon \mathrm{Ext}^1_{U_A}(\Omega_{U_A}, \mathcal{O}_U) \to \mathrm{Ext}^1_{\hat{U}_A}(\hat{\Omega}_{U_A}, \hat{\mathcal{O}}_U)$$

where $\hat{U}_A = \mathrm{Spec}(\hat{\mathcal{O}}_{U_A,P})$ and by $\hat{}$ we denote the completion at the closed point P. Let $\psi \colon \mathcal{O}_{U_A,P} \to \hat{\mathcal{O}}_{U_A,P}$ be the natural injective map. By a slight abuse of notation denote by z_i^A the images $\psi(z_i^A)$, $i = 1, \ldots, n$. Then there is $0 \le r(p) \le n$, such that

$$\hat{\mathcal{O}}_{U,P} = \frac{k[[z_0, \ldots, z_{r(p)}, t_{r(p)+1}, \ldots, t_n]]}{(z_0 \cdots z_n)}$$

and $z_i \in (\hat{\mathcal{O}}_{U,P})^*$, for $i > r(p)$. But now we are in the analytic case and the only logarithmic deformation over B is

$$\frac{B[[z_0, \ldots, z_{r(p)}, t_{r(p)+1}, \ldots, t_n]]}{(z_0 \cdots z_n - s_B)}$$

Therefore $\mu_P([U_B]) = \mu_P([\tilde{U}_B])$, for all $P \in U$, where $[U_B]$ and $[\tilde{U}_B]$ are the classes of U_B and $\tilde{U}_B$ in $\mathrm{Ext}^1_{U_A}(\Omega_{U_A}, \mathcal{O}_U)$. Hence $[U_B] = [\tilde{U}_B]$ and hence $U_B \cong \tilde{U}_B$ over U_A. Now the claim follows from Lemma 3.15.

□

The previous proposition shows that the local logarithmic deformation theory is very similar to the smooth one, the main similarity being the uniqueness of liftings. This allows us to work the log deformation theory with the same methods as the smooth. Repeating word by word the arguments of [KawNa94] we get the following theorem that was originally proved by Kawamata and Namikawa in the complex analytic case.

Theorem 3.17. *Let $(X, \mathcal{U}_0)$ be a normal crossing variety with a logarithmic structure on it. Let m be the number of irreducible components of X and $\Lambda_m = k[[x_1, \ldots, x_m]]$. Let*

$$0 \to J \to B \to A \to 0$$

be a square zero extension of local Artin Λ_m-algebras and let $(X_A, \mathcal{U}_A)$ be a logarithmic deformation of $(X, \mathcal{U})$ over A.

(1) *Assume that $(X_A, \mathcal{U}_A)$ lifts to a logarithmic deformation $(X_B, \mathcal{U}_B)$ over B. Then the set of automorphisms of X_B over B which fix the log structure $\mathcal{U}_B$ and induce the identity on X_A is bijective to*

$$H^0(X_A, T_{X_A/A}(log) \otimes_A J)$$

(2) *Under the same assumptions as in* (1)*, the set of equivalence classes of log deformations over B which are liftings of $(X_A, \mathcal{U}_A)$ is a torsor on*

$$H^1(X_A, T_{X_A/A}(log) \otimes_A J)$$

(3) *The obstruction to the existence of a lifting of $(X_A, \mathcal{U}_A)$ over B is in*

$$H^2(X_A, T_{X_A/A}(log) \otimes_A J)$$

Now exactly as in the case of usual deformations we have the following.

Theorem 3.18. *Let $(\mathcal{X}, \mathcal{U}_0)$ be a projective algebraic variety with normal crossing singularities with a logarithmic structure. Then the logarithmic deformation functor $LD(X, \mathcal{U}_0)$ has a hull* $\mathrm{LD}(X, \mathcal{U}_0)$ *in the sense of Schlessinger* [Sch68].

Finally, exactly as in [KawNa94, Corollary 2.4] we have the following.

Corollary 3.19. *Let $(X,\mathcal{U})$ be a normal crossing variety with a logarithmic structure. Then if $H^2(T_X(log))=0$, then* $\mathrm{LD}(X,\mathcal{U})$ *is smooth over Λ_m, where m is the number of irreducible components of the singular locus of X. Moreover, X is smoothable with a flat deformation $\mathcal{X}\to\Delta$, such that $\mathcal{X}$ is smooth.*

4. Obstructions

Let X be a variety with normal crossing singularities. It is well known that $H^2(T_X)$ and $H^1(T^1(X))$ are obstruction spaces to deformations of X. The purpose of this section is to describe these spaces and moreover to describe the obstruction space $H^2(T_X(log))$ to logarithmic deformations of a n.c. variety X with a logarithmic structure $\mathcal{U}$. We begin with some preliminary results.

Proposition 4.1 ([Fr83])**.** *Let X be a scheme with only normal crossing singularities. Let $\tau_X\subset\Omega_X$ be the torsion subsheaf of Ω_X. Then for all $i\geq 0$,*

$$\Omega_X/\tau_X\cong\Omega_X^{**}$$
$$\mathrm{Ext}^i_X(\Omega_X/\tau_X,\mathcal{O}_X)\cong H^i(T_X)$$
$$\mathrm{Ext}^i_X(\tau_X,\mathcal{O}_X)=H^{i-1}(T^1(X))$$

Corollary 4.2. *Let X be a projective scheme with only normal crossing singularities. Then*

$$H^2(T_X)=H^{n-2}((\Omega_X/\tau_X)\otimes\omega_X)$$

where $\dim X=n$.

Proof. By Proposition 4.1, $H^2(T_X)=\mathrm{Ext}^2_X(\Omega_X/\tau_X,\mathcal{O}_X)=H^{n-2}((\Omega_X/\tau_X)\otimes\omega_X)$, by Serre duality. □

Definition 4.3. Let X be a scheme with normal crossing singularities defined over a field k. Then we denote by $X_{[k]}\subset X$, $k\geq 0$, the subschemes of X defined inductively by $X_{[0]}=X$ and $X_{[k]}$ the singular locus of $X_{[k-1]}$ with reduced structure. We also denote by $\pi\colon\tilde{X}_{[i]}\to X_{[i]}$ the normalization of $X_{[i]}$ $i\geq 0$.

Theorem 4.4. *Let X be a scheme with normal crossing singularities defined over a field k. Then,*

(1) *There is an exact sequence*

(4.4.1)
$$0\to\Omega_X/\tau_X\to(\pi_0)_*\Omega_{\tilde{X}}\xrightarrow{\delta_1}(\pi_1)_*(\Omega_{\tilde{X}_{[1]}}\otimes L_1)\xrightarrow{\delta_2}\cdots\xrightarrow{\delta_N}(\pi_N)_*(\Omega_{\tilde{X}_{[N]}}\otimes L_N)\to 0$$

(2) *Suppose that X has a logarithmic structure $\mathcal{U}$. Then there is an exact sequence*

(4.4.2)
$$0\to\Omega_X/\tau_X\to\Omega_X(log)\xrightarrow{\lambda_1}(\pi_1)_*(\mathcal{O}_{\tilde{X}_{[1]}}\otimes M_1)\xrightarrow{\lambda_2}\cdots\xrightarrow{\lambda_m}(\pi_m)_*(\mathcal{O}_{\tilde{X}_{[m]}}\otimes M_m)\to 0$$

where $m,N\leq\dim X$ and L_i, M_j are 2-torsion invertible sheaves on $\tilde{X}_{[i]}$ and $\tilde{X}_{[j]}$, respectively, i.e., $L_i^{\otimes 2}\cong\mathcal{O}_{\tilde{X}_{[i]}}$ and $M_j^{\otimes 2}\cong\mathcal{O}_{\tilde{X}_{[j]}}$, for all i,j.

Remark 4.5. In the case of simple normal crossing complex analytic spaces, Theorem 4.4 was proved by R. Friedman [Fr83].

The following result is needed for the proof of the theorem.

Lemma 4.6. *Let $f\colon Y \to X$ be an étale morphism of schemes. Let $\pi\colon \tilde{X} \to X$ be the normalization of X. Then $p_Y\colon \tilde{X} \times_X Y \to Y$ is the normalization of Y, where $\tilde{X} \times_X Y \to Y$ is the fiber product of $\tilde{X}$ and Y over X and p_Y the projection to Y.*

Proof. From the fiber square diagram

$$\begin{tikzcd} \tilde{X} \times_X Y \arrow[r, "p_Y"] \arrow[d, "p_{\tilde{X}}"'] & Y \arrow[d, "f"] \\ \tilde{X} \arrow[r, "\pi"] & X \end{tikzcd}$$

it follows that $p_{\tilde{X}}$ is étale and p_Y finite. Hence $\tilde{X} \times_X Y$ is normal. Moreover p_Y is generically isomorphism. Therefore there is a factorization $g\colon \tilde{X} \times_X Y \to \tilde{Y}$ of p_Y through the normalization $\tilde{Y}$ of Y. But then, since both $\tilde{X} \times_X Y$, $\tilde{Y}$ are normal, g finite and generically isomorphism, g is in fact an isomorphism. $\square$

Proof of Theorem 4.4. We will only prove the first part in detail. The second part is similar and we will only sketch the proof and leave the details to the reader.

The proof of the first part is in two steps. First we show the existence of the exact sequence (4.4.1) for an affine simple normal crossing scheme and then we prove the general case. The proof of this part is similar to the one exhibited by Friedman [Fr83] in the case of a simple normal crossing complex analytic space. For the sake of completeness, and since the explicit local construction of the sequence is needed for the general case, we present a short proof here following the lines of Friedman's proof.

Step 1. Suppose

$$X = \mathrm{Spec} \frac{k[x_1, \ldots, x_n]}{(x_1 \cdots x_r)}$$

Then $X = \cup_{i=1}^r X_i$, where $X_i \subset X$ is the component given by $x_i = 0$, $1 \leq i \leq r$. Then for $i \geq 1$,

$$X_{[i]} = \bigcup_{k_0 < \cdots < k_i} (X_{k_0} \cap \cdots \cap X_{k_i})$$

Moreover, $\tilde{X} = \coprod_{i=1}^r X_i$ and

$$\tilde{X}_{[i]} = \coprod_{k_0 < \cdots < k_i} (X_{k_0} \cap \cdots \cap X_{k_i})$$

The maps $\pi_i\colon \tilde{X}_{[i]} \to X_{[i]}$, $i \geq 0$, are the natural ones. Now by definition, $\tau_X \subset \Omega_X$ is the sheaf of sections of Ω_X supported on the singular locus of X. Hence it is the kernel of the natural map $\delta\colon \Omega_X \to \pi_* \Omega_{\tilde{X}}$. Now define the sequence of maps
(4.6.1)

$$0 \to \tau_X \to \Omega_X \xrightarrow{\delta} (\pi_0)_* \Omega_{\tilde{X}} \xrightarrow{\delta_1} (\pi_1)_* \Omega_{\tilde{X}_{[1]}} \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_i} (\pi_i)_* \Omega_{\tilde{X}_{[i]}} \xrightarrow{\delta_{i+1}} (\pi_{i+1})_* \Omega_{\tilde{X}_{[i+1]}} \to \cdots$$

where δ_i are the Čech coboundary maps. This is clearly a complex and we proceed to show that it is in fact exact. We use induction on the number r of components of X. For $r = 1$ there is nothing to prove. Suppose now that the sequence (4.6.1) is exact for all simple normal crossing affine schemes with at most $r - 1$ components.

Let $X' = \cup_{i=1}^{r-1} X_i$ and $Y = X' \cap X_r$. Then, $X = X' \cup X_r$, $\tilde{X}_{[k]} = \tilde{X}'_{[k]} \coprod \tilde{Y}_{[k-1]}$, for all $k \geq 0$, where we also set $Y_{[-1]} = X_r$. By the induction hypothesis, the corresponding sequences (4.6.1) for X' and Y, are exact.

From the previous discussion, it follows that $\mathrm{Ker}(\delta) = \tau_X$. Next we show exactness at the next step, i.e., that $\mathrm{Ker}(\delta_1) = \mathrm{Im}(\delta)$. Now since

$$(\pi_0)_* \Omega_{\tilde{X}} = (\pi_0)_* \Omega_{\tilde{X}'} \oplus \Omega_{X_r}$$
$$(\pi_1)_* \Omega_{\tilde{X}_{[1]}} = (\pi_1)_* \Omega_{\tilde{X}'_{[1]}} \oplus \Omega_{\tilde{Y}}$$

any element of $(\pi_0)_* \Omega_{\tilde{X}}$ is of the form (α, β), where $\alpha \in (\pi_0)_* \Omega_{\tilde{X}'}$ and $\beta \in \Omega_{X_r}$. Suppose that such an element is also in the kernel of δ_1. It is now clear from the induction hypothesis that $(\alpha, 0)$ is in the image of δ. Therefore, in order to show exactness at the level of δ_1, it suffices to show that if an element of the form $(0, \beta)$ is in the kernel of δ_1, it is also in the image of δ. Suppose that $\beta = \sum_{k \neq r} \alpha_r(f_k) dx_k$ is such an element, where $f_k \in \mathcal{O}_X$ and $\alpha_r \colon \mathcal{O}_X \to \mathcal{O}_{X_r}$ the natural map. Therefore, since $(\pi_0)_* \Omega_{\tilde{Y}} = \oplus_{i=1}^{r-1} \Omega_{X_i \cap X_r}$, it follows that the restriction of β on $X_i \cap X_r$ is zero, for all $i \leq r-1$. Hence $f_k \in (x_1 \cdots \hat{x}_k \cdots x_{r-1}, x_r)$, for $1 \leq k \leq r$, and $f_k \in (x_1 \cdots x_{r-1}, x_r)$, for $k > r$. Therefore, $\delta(\sum_{k \neq r} f_k dx_k) = (0, \beta)$ and hence $(0, \beta)$ is in the image of δ.

There is an exact sequence

$$0 \to (\pi_{k-1})_* \Omega_{\tilde{Y}_{[k-1]}} \to (\pi_k)_* \Omega_{\tilde{X}_{[k]}} \to (\pi_k)_* \Omega_{\tilde{X}'_{[k]}} \to 0 \tag{4.6.2}$$

Now define the complexes (A^*, δ_A^*), (B^*, δ_B^*) and (C^*, δ_C^*), such that $A^k = (\pi_{k-1})_* \Omega_{\tilde{Y}_{[k-1]}}$, $B^k = (\pi_k)_* \Omega_{\tilde{X}_{[k]}}$, and $C^k = (\pi_k)_* \Omega_{\tilde{X}'_{[k]}}$, $k \geq 0$, and the coboundary maps are the Čech maps. Then (4.6.2) induces an exact sequence of complexes

$$0 \to A^* \to B^* \to C^* \to 0$$

Passing to cohomology we get an exact sequence

$$\cdots \to H^k(A^*) \to H^k(B^*) \to H^k(C^*) \to \cdots$$

Now by induction $H^k(C^*) = 0$, for all $k \geq 1$ and $H^k(A^*) = 0$, for all $k \geq 2$. Hence $H^k(B^*) = 0$, for all $k \geq 2$. It remains to check for $k = 1$. Then there is an exact sequence

$$H^0(C^*) \xrightarrow{\sigma} H^1(A^*) \to H^1(B^*) \to 0$$

Moreover, $H^0(C^*) = \Omega_{X'}/\tau_{X'}$, $H^1(A^*) = \Omega_Y/\tau_Y$ and σ is the natural map

$$\Omega_{X'}/\tau_{X'} \to \Omega_Y/\tau_Y$$

and hence it is surjective. Therefore $H^1(B^*) = 0$ and the complex (4.6.1) is exact as claimed.

Step 2. The general case. So, let X be a scheme with normal crossing singularities.

Claim: There is an étale cover $(U_i \xrightarrow{f_i} X)_{i \in I}$ of X such that

(1) U_i is a simple normal crossing scheme.
(2) For any $i \in I$, all irreducible components of U_i pass through the same point.
(3) Let $U_i = \cup_{k=1}^{N_i} U_{i,k}$ be the decomposition of U_i into its irreducible components. Then there is an exact sequence

(4.6.3)
$$0 \to \tau_{U_i} \to \Omega_{U_i} \xrightarrow{\delta} (\pi_{i,0})_* \Omega_{\tilde{U}_i} \xrightarrow{\delta_{i,1}} (\pi_{i,1})_* \Omega_{\tilde{U}_{i,[1]}} \xrightarrow{\delta_{i,2}} \cdots \xrightarrow{\delta_{i,k}} (\pi_{i,k})_* \Omega_{\tilde{U}_{i,[k]}} \xrightarrow{\delta_{i,k+1}} \cdots$$

where

$$U_{i,[k]} = (U_i)_{[k]} = \bigcup_{s_0 < \cdots < s_k} (U_{i,s_0} \cap \cdots \cap U_{i,s_k})$$

and the boundary maps are the Čech maps.

We proceed to show the claim. Let $P \in X$ a point. Then by assumption,

$$\hat{\mathcal{O}}_{X,P} \cong \frac{k[[x_1, \ldots, x_n]]}{(x_1 \cdots x_{r(p)})}$$

Let $W_P = \operatorname{Spec}(k[x_1, \ldots, x_n]/(x_1 \cdots x_{r(p)}))$ and let w_p be the closed point corresponding to the maximal ideal $(x_1, \ldots, x_n)$. Then by Theorem 3.3, since $\hat{\mathcal{O}}_{X,P} \cong \hat{\mathcal{O}}_{W_P,w_p}$, there is a common étale neighborhood of $P \in X$ and $w_p \in W_P$, i.e., there are étale maps

$$\begin{tikzcd} & U_P \arrow[dl, "f_p"'] \arrow[dr, "g_p"] & \\ X & & W_P \end{tikzcd}$$

and a point $u_p \in U_P$ such that $f_p(u_p) = P$, $g_p(u_p) = w_p$ and inducing isomorphisms of residue fields $k(u_p) \cong k(P) \cong k(w_p)$. Let $W_p = \cup_{i=1}^{r(p)} W_{p,i}$ its decomposition into irreducible components, where $W_{p,i}$ is given by $x_i = 0$, $i = 1, \ldots, r(p)$. Then $U_p = \cup_{i=1}^{r(p)} g_p^{-1}(W_{p,i})$. Moreover, since g_p is étale, $g_p^{-1}(W_{p,i})$ is smooth (perhaps disconnected) and therefore U_p is a simple normal crossing variety. Taking a sufficiently small neighborhood of $u_p \in U_P$, we may assume that all irreducible components of U_P pass through u_p. Then from step 1. there is an exact sequence

$$0 \to \tau_{W_P} \to \Omega_{W_P} \stackrel{\delta}{\to} (\pi_{w_p,0})_* \Omega_{\tilde{W_P}} \stackrel{\delta_{w_p,1}}{\to} (\pi_{w_p,1})_* \Omega_{\tilde{W}_{P,[1]}} \stackrel{\delta_{w_p,2}}{\to} \cdots \tag{4.6.4}$$

where for convenience we set $W_{p,[k]} = (W_P)_{[k]}$. Since g_p is étale, by Lemma 4.6 there is a fiber square diagram

$$\begin{tikzcd} \tilde{U}_{P,[k]} \arrow[r, "\pi_{u_p,k}"] \arrow[d, "\tilde{g_p}"'] & U_{P,[k]} \arrow[d, "g_p"] \\ \tilde{W}_{P,[k]} \arrow[r, "\pi_{w_p,k}"] & W_{P,[k]} \end{tikzcd}$$

By flat base change it follows that $g_p^*(\pi_{w_p,k})_* = (\pi_{u_p,k})_*(\tilde{g}_p)^*$. Moreover, since both g_p and $\tilde{g}_p$ are étale,

$$g_p^* \Omega_{W_{p,[k]}} = \Omega_{U_{p,[k]}}$$
$$(\tilde{g}_p)^* \Omega_{\tilde{W}_{p,[k]}} = \Omega_{\tilde{U}_{p,[k]}}$$

Therefore (4.6.4) pulls back via g_p to an exact sequence in U_p

$$0 \to \tau_{U_P} \to \Omega_{U_P} \stackrel{\delta}{\to} (\pi_{u_p,0})_* \Omega_{\tilde{U_P}} \stackrel{\delta_{u_p,1}}{\to} (\pi_{u_p,1})_* \Omega_{\tilde{U}_{P,[1]}} \stackrel{\delta_{u_p,2}}{\to} \cdots \tag{4.6.5}$$

and the coboundary maps are the Čech maps corresponding to the numbering of the components of U_P induced from the numbering of the components of W_P. R epeating this procedure at all points of X, we obtain an étale cover $(U_i \stackrel{f_i}{\to} X)_{i \in I}$ with the properties claimed.

Let $V_i = f_i(U_i) \subset X$. Then since f_i is étale, V_i is a zariski open set of X. Next we show that the exact sequences (4.6.3) on U_i are induced by analogous exact sequences on V_i. Let $\Omega^{et}_{V_i}$, $\Omega^{et}_{\tilde{V}_{i,[k]}}$ be the sheaves in the étale topologies of V_i and $\tilde{V}_{i,[k]}$ induced by Ω_{V_i}, $\Omega_{\tilde{V}_{i,[k]}}$. They have the property that for any étale map $Z \to V_i$, $\Omega^{et}_{V_i}|_Z = \Omega_Z$, and similarly for $\Omega^{et}_{\tilde{V}_{i,[k]}}$ [Mil80]. Then the maps $\delta_{i,k}$ induce maps in the étale topology

$$(\pi_{i,k})_*\Omega^{et}_{\tilde{U}_{i,[k]}} \xrightarrow{\delta^{et}_{i,k}} (\pi_{i,k+1})_*\Omega^{et}_{\tilde{U}_{i,[k+1]}}$$

Then since U_i is an étale cover of V_i, the necessary and sufficient condition for $\delta^{et}_{i,k}$ to be induced by a map

$$\delta^{et}_{i,k} \colon (\pi_{i,k})_*\Omega^{et}_{\tilde{V}_{i,[k]}} \to (\pi_{i,k+1})_*\Omega^{et}_{\tilde{V}_{i,[k+1]}}$$

is that

$$p_1^*(\delta^{et}_{i,k}) = p_2^*(\delta^{et}_{i,k})$$

where p_1, p_2 are the projection maps from the fiber square diagram

$$\begin{tikzcd} U_i \times_{V_i} U_i \arrow[r, "p_2"] \arrow[d, "p_1"'] & U_i \arrow[d, "f_i"] \\ U_i \arrow[r, "f_i"] & V_i \end{tikzcd}$$

Let $Z_i = U_i \times_{V_i} U_i$. Since p_i are étale, Z_i is a simple normal crossing variety and

$$Z_i = \cup_{k=1}^{N_i} p_1^{-1}(U_{i,k}) = \cup_{k=1}^{N_i} p_2^{-1}(U_{i,k})$$

is its decomposition into smooth irreducible components. Then $p_1^*(\delta^{et}_{i,k})$, $p_2^*(\delta^{et}_{i,k})$ are Čech maps

$$(\pi_{i,k})_*\Omega^{et}_{\tilde{Z}_{i,[k]}} \to (\pi_{i,k+1})_*\Omega^{et}_{\tilde{Z}_{i,[k+1]}}$$

corresponding to the numberings of the irreducible components of Z_i induced by p_1 and p_2, respectively. Hence

$$p_1^*(\delta^{et}_{i,k}) = \epsilon p_2^*(\delta^{et}_{i,k})$$

where $\epsilon \in \{1, -1\}$. I now claim that the numbering of the components of Z_i induced by p_1 is the same as the numbering induced by p_2. To show this it suffices to show that $p_1^{-1}(U_{i,k}) = p_2^{-1}(U_{i,k})$, for all i. Suppose that this is not true. Then there is an irreducible component, say $Z_{i,k}$, of $p_1^{-1}(U_{i,k})$ (since p_1 is étale, $p_1^{-1}(U_{i,k})$ is a disjoint union of smooth irreducible varieties) such that $p_2(Z_{i,k}) = U_{i,\lambda}$, with $k \neq \lambda$. By the construction to the étale cover, $u_i \in U_{i,k} \cap U_{i,\lambda}$. Let z_i, such that $p_1(z_i) = p_2(z_i) = u_i$ and $v_i = f(u_i)$. Then by passing to completions we get a commutative diagram

$$\begin{tikzcd} & \hat{\mathcal{O}}_{Z_i,z_i} \arrow[dl, "\hat{p}_1"'] \arrow[dr, "\hat{p}_2"] & \\ \hat{\mathcal{O}}_{U_i,u_i} \arrow[dr, "\hat{f}"'] & & \hat{\mathcal{O}}_{U_i,u_i} \arrow[dl, "\hat{f}"] \\ & \hat{\mathcal{O}}_{V_i,v_i} & \end{tikzcd}$$

Since p_1, p_2 and f are étale, it follows that $\hat{p}_1$, $\hat{p}_2$ and $\hat{f}$ are isomorphisms and since we are assuming that X has normal crossing singularities, all rings involved are isomorphic to $k[[x_1,\dots,x_n]]/(x_1\cdots x_s)$, for some $1\leq s\leq n$. Then, if there is an i such that $p_1^{-1}(U_{i,k})\neq p_2^{-1}(U_{i,k})$, there are two different components of $\hat{\mathcal{O}}_{U_i,u_i}$, corresponding to $U_{i,k}$ and $U_{i,\lambda}$ that map to the same component of $\hat{\mathcal{O}}_{V_i,v_i}$, which is impossible. Hence the maps $\delta_{i,k}^{et}$ are induced by maps on V_i. This means that for each i there is sequence of maps
(4.6.6)

$$0\to\tau_{V_i}\to\Omega_{V_i}\overset{\delta}{\to}(\pi_{i,0})_*\Omega_{\tilde{V}_i}\overset{\delta_{i,1}}{\to}(\pi_{i,1})_*\Omega_{\tilde{V}_{i,[1]}}\overset{\delta_{i,2}}{\to}\cdots\overset{\delta_{i,k}}{\to}(\pi_{i,k})_*\Omega_{\tilde{V}_{i,[k]}}\overset{\delta_{i,k+1}}{\to}\cdots$$

which pulls pack by f_i^* to (4.6.3). Since (4.6.3) is exact and $f_i\colon U_i\to V_i$ faithfully flat, (4.6.6) is exact too.

Next we glue the maps $\delta_{i,k}$ to obtain global maps δ_k. To this we check how they differ on overlaps. Let i,j with $i\neq j$. Let $f_i\colon U_i\to V_i$ and $f_j\colon U_j\to V_j$ the corresponding étale cover. Let $V_{ij}=V_i\cap V_j$ and once again consider the following fiber square diagram

$$\begin{tikzcd} Z_{ij} \arrow[r, "p_2"] \arrow[d, "p_1"'] & f_i^{-1}(V_{ij}) \arrow[d, "f_i"] \\ f_j^{-1}(V_{ij}) \arrow[r, "f_j"] & V_{ij} \end{tikzcd}$$

where $Z_{ij}=f_i^{-1}(V_{ij})\times_{V_{ij}}f_j^{-1}(V_{ij})$. Then $f_i^{-1}(V_{ij})$, $f_j^{-1}(V_{ij})$ and Z_{ij} are simple normal crossing varieties. Let $h_1=f_jp_1$, $h_2=f_ip_2$. Then h_1 and h_2 are étale. Since they are also faithfully flat, it suffices to compare $h_1^*\delta_{i,k}$ and $h_2^*\delta_{j,k}$. By construction, both of these maps are Čech coboundary maps between $(\pi_{ij,k})_*\Omega_{\tilde{Z}_{ij,[k]}}$ and $(\pi_{ij,k+1})_*\Omega_{\tilde{Z}_{ij,[k+1]}}$ corresponding to two (perhaps different) numberings of the components of Z_{ij}. Hence $h_1^*\delta_{i,k}=\varepsilon_ih_2^*\delta_{j,k}$, $\varepsilon_i\in\{-1,1\}$. Therefore $\delta_{i,k}=\varepsilon_id_{j,k}$. Therefore, in order to be able to glue them we change the glueing maps of $\Omega_{\tilde{V}_{i,[k]}}$ by multiplying the canonical ones by ε_i. This amounts into tensoring $\Omega_{\tilde{X}_{[k]}}$ by a 2-torsion sheaf and hence we get a global sequence

$$0\to\tau_X\to\Omega_X\to(\pi_0)_*\Omega_{\tilde{X}}\overset{\delta_1}{\to}(\pi_1)_*(\Omega_{\tilde{X}_{[1]}}\otimes L_1)\overset{\delta_2}{\to}(\pi_2)_*(\Omega_{\tilde{X}_{[2]}}\otimes L_2)\to\cdots$$

where $L_i\in\operatorname{Pic}(\tilde{X}_{[k]})$ are 2-torsion invertible sheaves as claimed.

Next we sketch the proof of the second part. Initially we construct the exact sequence (4.4.2) in the case when

$$X=\operatorname{Spec}\frac{k[x_0,\dots,x_n]}{(x_0\cdots x_r)}$$

In this case $X=\cup_{i=1}^rX_r$, where X_i is given by $x_i=0$. Then,

$$(\pi_i)_*\mathcal{O}_{\tilde{X}_i}=\bigoplus_{j_0<\cdots<j_i}\mathcal{O}_{X_{j_0}\cap\cdots\cap X_{j_i}}$$

Then we define the sequence of maps
(4.6.7)

$$0\to\tau_X\to\Omega_X\overset{\lambda_0}{\to}\Omega_X(log)\overset{\lambda_1}{\to}(\pi_1)_*\mathcal{O}_{\tilde{X}_{[1]}}\overset{\lambda_2}{\to}\cdots\overset{\lambda_i}{\to}(\pi_i)_*\mathcal{O}_{\tilde{X}_{[i]}}\overset{\lambda_{i+1}}{\to}(\pi_{i+1})_*\mathcal{O}_{\tilde{X}_{[i+1]}}\to\cdots$$

as follows. The maps λ_i are the Čech coboundary maps for $i\geq 2$ and λ_0 is the natural map between Ω_X and $\Omega_X(log)$. $\Omega_X(log)$ is a free $\mathcal{O}_X$-module generated

by $dx_1/x_1, \dots, dx_r/x_r, dx_{r+1}, \dots, dx_n$ with the relation $dx_0/x_0 + \cdots dx_r/x_r = 0$. Then we define $\lambda_1(dx_i) = 0$, if $i > r$ and if $i \le r$, $\lambda_1(dx_i/x_i) = (\alpha_{j_0,j_1}), j_0 < j_1$ such that

$$\alpha_{j_0,j_1} = \begin{cases} 1 & \text{if } j_0 = i \\ -1 & \text{if } j_1 = i \\ 0 & \text{otherwise} \end{cases}$$

Now by either using the same method as in the first part for the sequence (4.4.1) or by [Fr83, Corollary 3.6], we get that (4.6.7) is exact. Now by using exactly the same argument as in the first part by using étale covers we get the existence of (4.4.2). □

Theorem 4.7. *Let X be a Fano variety with normal crossing singularities defined over a field k of characteristic zero. Then $H^2(T_X) = 0$. Moreover, if X has a logarithmic structure $\mathcal{U}$, then $H^2(T_X(log)) = 0$ as well.*

The previous theorem and Theorem 3.18 imply that

Corollary 4.8. *Let X be a Fano variety with normal crossing singularities defined over a field k of characteristic zero. Assume that X has a logarithmic structure $\mathcal{U}$. Then $\mathrm{LD}(X,\mathcal{U})$ is smooth.*

Proof of Theorem 4.7. By Corollary 4.2,

$$H^2(T_X) = H^{n-2}((\Omega_X/\tau_X) \otimes \omega_X) \tag{4.8.1}$$

where $n = \dim X$. Then by Theorem 4.4, there is an exact sequence

(4.8.2)

$$0 \to \Omega_X/\tau_X \xrightarrow{\delta_0} (\pi_0)_*\Omega_{\tilde{X}} \xrightarrow{\delta_1} (\pi_1)_*(\Omega_{\tilde{X}_{[1]}} \otimes L_1) \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_N} (\pi_N)_*(\Omega_{\tilde{X}_{[N]}} \otimes L_N) \to 0$$

where $N \le \dim X$, τ_X is the torsion part of Ω_X and L_i is an invertible sheaf on $\tilde{X}_{[i]}$ such that $L_i^{\otimes 2} \cong \mathcal{O}_{\tilde{X}_{[i]}}$.

Tensoring (4.8.2) with ω_X and taking into consideration that

$$(\pi_i)_*(\Omega_{\tilde{X}_{[i]}} \otimes (\pi_i)^*\omega_X) = (\pi_i)_*\Omega_{\tilde{X}_{[i]}} \otimes \omega_X$$

we get the exact sequence

$$0 \to (\Omega_X/\tau_X) \otimes \omega_X \xrightarrow{\delta_0} (\pi_0)_*(\Omega_{\tilde{X}} \otimes \pi_0^*\omega_X) \xrightarrow{\delta_1} (\pi_1)_*(\Omega_{\tilde{X}_{[1]}} \otimes L_1 \otimes \pi_1^*\omega_X) \xrightarrow{\delta_2} \cdots$$
$$\cdots \xrightarrow{\delta_N} (\pi_N)_*(\Omega_{\tilde{X}_{[N]}} \otimes L_N \otimes \pi_N^*\omega_X) \to 0$$

Let $M_k = \mathrm{Im}(\delta_k)$, $1 \le k \le N$. Then the above sequence splits into

$$0 \to (\Omega_X/\tau_X) \otimes \omega_X \xrightarrow{\delta_1} (\pi_0)_*(\Omega_{\tilde{X}} \otimes \pi_0^*\omega_X) \xrightarrow{\delta_2} M_1 \to 0$$
$$0 \to M_k \to (\pi_k)_*(\Omega_{\tilde{X}_{[k]}} \otimes L_k \otimes \pi_k^*\omega_X) \to M_{k+1} \to 0$$

where $1 \le k \le N-1$, $N \le n = \dim X$ and $M_N = (\pi_N)_*(\Omega_{\tilde{X}_{[N]}} \otimes L_N \otimes \pi_N^*\omega_X)$. Therefore we get exact sequences in cohomology

(4.8.3)

$$\cdots H^{n-3}(M_1) \to H^{n-2}((\Omega_X/\tau_X) \otimes \omega_X) \to H^{n-2}((\pi_0)_*(\Omega_{\tilde{X}} \otimes \pi_0^*\omega_X)) \to \cdots$$
$$\cdots H^{n-k-3}(M_{k+1}) \to H^{n-k-2}(M_k) \to H^{n-k-2}((\pi_k)_*(\Omega_{\tilde{X}_{[k]}} \otimes L_k \otimes \pi_k^*\omega_X)) \to \cdots$$

Now since π_k are finite, it follows that $(\pi_k^*\omega_X)^{-1}$ are ample, for all $0 \leq k \leq N$ and hence $(L_k^{-1} \otimes \pi_k^*\omega_X)^{-1}$ is ample too, since L_k is 2-torsion and invertible. Moreover, $\tilde{X}_{[k]}$ is smooth of dimension $n-k$. Therefore, and by using the Kodaira-Nakano vanishing theorem [EV92, Corollary 6.4],

$$H^{n-k-2}((\pi_k)_*(\Omega_{\tilde{X}_{[k]}} \otimes L_k \otimes \pi_k^*\omega_X)) = 0$$

for all $1 \leq k \leq N$. Hence from (4.8.3) and by induction it follows that $H^{n-k-2}(M_k) = 0$, for all $0 \leq k \leq N$ and hence again by (4.8.3) it follows that there is an exact sequence

$$H^{n-3}(M_1) \to H^{n-2}((\Omega_X/\tau_X) \otimes \omega_X) \to H^{n-2}((\pi_0)_*(\Omega_{\tilde{X}} \otimes \pi_0^*\omega_X))$$

and therefore

$$H^2(T_X) = H^{n-2}((\Omega_X/\tau_X) \otimes \omega_X) = 0$$

as claimed.

If X has a logarithmic structure $\mathcal{U}$, then by using the exact sequence (4.4.2) and arguing similarly as before we get that $H^2(T_X(log)) = 0$. □

Unfortunately, in general I cannot say much about the other obstruction space, namely $H^1(T^1(X))$. However, since $T^1(X)$ is a line bundle on the singular locus $X_{[1]}$ of X, it is much easier handled than $H^2(T_X)$ and it will vanish if we impose certain positivity requirements on $T^1(X)$.

The case when X has only double points exhibits much better behavior and it deserves special consideration. The difference between this and the general case is that the singular locus $X_{[1]}$ of X is smooth.

Theorem 4.9. *Let X be a Fano variety with only double point normal crossing singularities such that $T^1(X)$ is finitely generated by its global sections. Then*

$$H^2(T_X) = H^1(T^1(X)) = 0$$

Corollary 4.10. *Let X be a Fano variety with only double point normal crossing singularities and such that $T^1(X)$ is finitely generated by its global sections. Then* $\mathrm{Def}(X)$ *is smooth.*

Question 4.11. Is $Def(X)$ smooth for any Fano variety with normal crossing singularities? If this is true then X is smoothable if and only if $T^1(X)$ is finitely generated by global sections and hence this is a very natural condition to impose.

Remark 4.12. In general, $H^1(T^1(X))$ will not vanish. However if X is smoothable, then $T^1(X)$ must have some positivity properties and the one stated is the most natural one.

Proof of Theorem 4.9. In view of Theorem 4.7 we only need to show the vanishing of $H^1(T^1(X))$. In order to show this we will first show that the singular locus $Z = X_{[1]}$ of X is a smooth Fano variety of dimension $\dim X - 1$. The Fano part is the only part to be shown. Let $\pi\colon \tilde{X} \to X$ be the normalization and $\tilde{Z} = \pi^{-1}Z$. Then $\tilde{Z} \to Z$ is étale. By subadjunction we get that

$$\pi^*\omega_X = \omega_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(\tilde{Z})$$

Therefore,

$$\omega_{\tilde{Z}} = \omega_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(\tilde{Z}) \otimes \mathcal{O}_{\tilde{Z}}$$

Hence $\omega_{\tilde{Z}}^{-1}$ is ample. But since $\tilde{Z} \to Z$ is étale, it follows that $\pi^*\omega_Z = \omega_{\tilde{Z}}$. Therefore ω_Z^{-1} is ample too and hence Z is Fano as claimed. Now

$$H^1(T^1(X)) = H^1(\omega_Z \otimes (T^1(X) \otimes \omega_Z^{-1})) = 0$$

by the Kawamata-Viehweg vanishing theorem since if $T^1(X)$ is finitely generated by global sections, then $T^1(X) \otimes \omega_Z^{-1}$ is ample too.

□

5. Smoothings of Fanos

In this section we obtain criteria for a Fano variety X with normal crossing singularities to be smoothable. First we state a criterion for a variety X with hypersurface singularities to be smoothable and moreover to be smoothable with a smooth total space.

Proposition 5.1. *Let X be a reduced projective scheme with hypersurface singularities and let D be its singular locus. Then*

1. *If X is smoothable by a flat deformation $\mathcal{X} \to \Delta$ such that $\mathcal{X}$ is smooth, then $T^1(X) = \mathcal{O}_D$.*
2. *Suppose that $T^1(X)$ is finitely generated by its global sections and that $H^2(T_X) = H^1(T^1(X)) = 0$. Then X is smoothable. Moreover, if $Def(X)$ is smooth then the converse is also true.*

Proof. The second part is [Tz10, Theorem 12.5]. We proceed to show the first part. Let $f\colon \mathcal{X} \to \Delta$ be a smoothing of X such that $\mathcal{X}$ is smooth, where $\Delta = \mathrm{Spec}(R)$, (R, m_R) is a discrete valuation ring. Let $T^1(\mathcal{X}/\Delta) = \mathcal{E}xt^1_{\mathcal{X}}(\Omega_{\mathcal{X}/\Delta}, \mathcal{O}_{\mathcal{X}})$ be Schlessinger's relative T^1 sheaf. Then dualizing the exact sequence

$$0 \to f^*\omega_\Delta = \mathcal{O}_{\mathcal{X}} \to \Omega_{\mathcal{X}} \to \Omega_{\mathcal{X}/\Delta} \to 0$$

we get the exact sequence

$$\cdots \to \mathcal{O}_{\mathcal{X}} \to T^1(\mathcal{X}/\Delta) \to \mathcal{E}xt^1_{\mathcal{X}}(\Omega_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}) = 0$$

Now restricting to the special fiber and taking into consideration that $\mathcal{X} \otimes_R R/m_r \cong X$ and that $T^1(\mathcal{X}/\Delta) \otimes_R R/m_R = T^1(X)$ [Tz10, Lemma 7.7], we get that there is a surjection $\mathcal{O}_X \to T^1(X)$. Moreover, $T^1(X)$ is a line bundle on the singular locus D of X. Hence, restricting on Z it follows that $T^1(X) \cong \mathcal{O}_D$, as claimed. □

Remark 5.2. The condition $T^1(X) = \mathcal{O}_D$ is equivalent to Friedman's d-semistability condition in the case of reducible simple normal crossing schemes [Fr83]. One of the natural questions raised by Friedman is whether this condition is sufficient for a simple normal crossing variety to be smoothable. He showed that in the case of $K3$ surfaces this is true but Persson and Pinkham have shown that this is not true in general [PiPe83]. However this is true in the case of normal crossing (not necessarily reducible) Fano schemes, as shown by Theorem 5.3 below.

Theorem 5.3. *Let X be a Fano variety defined over an algebraically closed field of characteristic zero with normal crossing singularities. Assume that one of the following conditions hold:*

1. *$T^1(X)$ is finitely generated by global sections and that $H^1(T^1(X)) = 0$.*
2. *X has at worst double point normal crossing singularities and that $T^1(X)$ is finitely generated by global sections.*

(3) *X is d-semistable, i.e., $T^1(X) \cong \mathcal{O}_D$, where D is the singular locus of X.*

Then X is smoothable. Moreover, X is smoothable by a flat deformation $f\colon \mathcal{X} \to \Delta$ such that $\mathcal{X}$ is smooth, if and only if X is d-semistable.

Proof. (5.3.1) follows directly from Theorem 4.7 and Proposition 5.1.2. (5.3.2) follows from Theorems 4.9 and 5.1.2. Finally, suppose that $T^1(X) \cong \mathcal{O}_D$, i.e., X is d-semistable. Then according to Proposition 3.8, X admits a logarithmic structure $\mathcal{U}$. Hence by Corollary 3.19, $(X, \mathcal{U})$ has unobstructed logarithmic deformations and is smoothable. □

6. Examples.

In this section we construct one example of a smoothable and one of a non-smoothable normal crossing Fano 3-fold.

Example 6.1. Let $P \in Y \subset \mathbb{P}^4$ be a quadric surface with one ordinary double point locally analytically isomorphic to $(xy - zw = 0) \subset \mathbb{C}^4$. Let $f\colon X \to Y$ be the blow up of $P \in Y$. Then X is smooth and the f-exceptional divisor E is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover, $-K_X - E$ is ample and $\mathcal{N}_{E/X} = \mathcal{O}_E(-1,-1)$.

Next we construct an embedding $E \subset X'$ of E into a smooth scheme X' such that, $\mathcal{N}_{E/X'} = \mathcal{O}_E(1,1)$ and $-K_{X'} - E$ is ample. Let $Z \subset \mathbb{P}^3$ be a smooth quadric surface. Then $\mathcal{N}_{Z/\mathbb{P}^3} = \mathcal{O}_Z(2,2)$. Let $\pi\colon X' \to \mathbb{P}^3$ be the cyclic double cover of $\mathbb{P}^3$ ramified over Z. This is defined by the line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}^3}(1)$ and the section s of $\mathcal{L}^{\otimes 2}$ that corresponds to Z. Let $E = (\pi^{-1}(Z))_{\mathrm{red}} \cong Z$. Then $\pi^* Z = 2E$ and $\omega_X = \pi^*(\omega_{\mathbb{P}^3} \otimes \mathcal{L})$. Let $l' \subset E$ be one of the rulings and $l = \pi_*(l')$. then

$$l' \cdot E = 1/2(l' \cdot \pi^* Z) = 1/2(l \cdot Z) = 1$$

and hence $\mathcal{N}_{E/\mathcal{X}'} = \mathcal{O}_E(1,1)$. Now let Y be the scheme obtained by glueing X and X' along E. This is a normal crossing Fano 3-fold with only double points. Then by Theorem 3.6, $T^1(Y) = \mathcal{N}_{E/X} \otimes \mathcal{N}_{E/X'} = \mathcal{O}_E$. Therefore, by Theorem 5.3, Y is smoothable.

Example 6.2. Let $E \subset X$ be as in example 1. Then let Y be obtained by glueing two copies of X along E. Then by Theorem 3.6,

$$T^1(Y) = \mathcal{N}_{E/X} \otimes \mathcal{N}_{E/X} = \mathcal{O}_E(-2,-2)$$

and hence $H^0(T^1(Y)) = 0$. Hence Y is not smoothable [Tz09, Theorem 12.3]. In fact, every deformation of Y is locally trivial.

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